New Existence Theorems about the Solutions of Some Stochastic Integral Equations

Xuemei Chen and Yingying Qi and Chunyan Yang
Department of Mathematics, Sichuan University,
Chengdu 610064, P.R.China

Abstract Picard's iteration has been used to prove the existence and uniqueness of the solution for stochastic integral equations, here we use Schauder's fixed point theorem to give a new existence theorem about the solution of a stochastic integral equation, our theorem can weak some conditions gotten by applying Banach's fixed point theorem.

Keywords Stochastic integral equation; Schauder's fixed point theorem; bounded closed convex subset; compact operator

AMS Subject Classification 47H10, 60H05, 60H10

1 Introduction and Main Results

Many mathematical models of phenomena occuring in sociology, physics, biology and engineering involve random differential and integral equations. Theoretical and applied treatments of problems concerning random differential and integral equations can be found in many papers and monographs: Bharucha-Reid ([4]); Doob ([5]); Padgeett and Tsokos ([6]); Tsokos and Padgett ([7]); Rao and Tsokos ([8]).

For stochastic differential equations, firstly we should prove the existence for the solutions. In filtering problem we know the system X_t satisfying $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB(t)$, the observations Z_s satisfying $dZ_s = c(s, X_s)ds + d(s, X_s)dV(s), Z_0 = 0$ for $0 \leq s \leq t$, where V(s) is Brownian motion, we want to find the best estimate $\hat{X}_t$ of the state X_t of the system based on these observations, before we find the estimate $\hat{X}_t$, we should give some assumptions for the existence of the corresponding stochastic integral equations.

Some mathematicians have used Picard's iteration or Banach's contraction mapping principle to prove the existence and uniqueness of some stochastic integral equations. The goal of this paper is to give a new existence theorem about stochastic integral equations using Schauder's fixed point theorem, in order to apply Schauder's fixed point theorem we need to construct a compact operator A and a convex bounded and closed nonempty subset M. Furthermore, comparing with Banach's fixed point theorem we weak some conditions.

Theorem 1.1([1])(Schauder's fixed point theorem). *The compact operator*

$$A : M \longrightarrow M$$

has at least one fixed point when M is a bounded, closed, convex, nonempty subset of a Banach space X over real field.

Theorem 1.2([1])(Banach's fixed point theorem). *We assume that:*

(a) M is a closed nonempty subset in the Banach space X over field K, and

(b) the operator $A : M \longrightarrow M$ is k-contractive, i.e., there is $0 \le k < 1$ such that:

$$\|Au - Av\| \le k\|u - v\| \quad for\ all\ u, v \in M.$$

Then, the operator A has exactly one fixed point u on the set M.

Schauder's Fixed Point Theorem can be applied to many fields in mathematics, especially to the integral equation:

$$u(x) = \lambda \int_a^b F(x, y, u(y))dy, \quad a \le x \le b, \tag{1}$$

where $-\infty < a < b < +\infty$ and $\lambda \in R$. Let

$$Q = \{(x, y, u) \in R^3;\ x, y \in [a, b],\ |u| \le r\} \quad for\ fixed\ r > 0.$$

Theorem 1.3([1])*Assume the following conditions:*

(a) The function $F : Q \longrightarrow R$ is continuous.

(b) We define $(b - a)\mathcal{M} := \max_{(x,y,u)\in Q} \ |F(x, y, u)|$, let the real number λ be given such that

$$|\lambda|\mathcal{M} \le r \tag{2}$$

(c) We set $X := C[a, b]$ and $M := \{u \in X; \|u\| \le r\}$.

Then the original integral equation (1) has at least one solution $u \in M$.

It well known that Banach's Fixed Point Theorem can be used to prove the existence and uniqueness of the solution for the integral equation(1).

Theorem 1.4([1])*Assume the following conditions:*

(a) The function $F : [a, b] \times [a, b] \times R \longrightarrow R$ is continuous, and the partial derivative

$$F_u : [a, b] \times [a, b] \times R \longrightarrow R$$

is also continuous.

(b) There is a number $\mathcal{L}$ such that

$$|F_u(x, y, u)| \le \mathcal{L} \quad for\ all\ x, y \in [a, b], u \in R.$$

(c) Let the real number λ be given such that

$$(b - a)|\lambda|\mathcal{L} < 1. \tag{3}$$

(d) Set $X := C[a,b]$ and $\|u\| := max_{a\le x\le b}|u(x)|$.

Then the original equation (1) has a unique solution $u \in M$.

From Theorem 1.1 and Theorem 1.2 we know that Schauder's fixed point theorem is a existence principle, while Banach's fixed point theorem is a existence and uniqueness theorem. It seems that the conditions in Theorem 1.3 are weaker comparing with Theorem 1.4, that is, (2) is easier to reach than (3).

Here, a nature question is whether Schauder's and Banach's fixed point theorems can be applied to stochastic integral equations. Furthermore, whether the conditions coming from Banach's fixed point theorem are stronger than the conditions from Schauder's fixed point theorem.

Notation 2.1([2]) *For convenience, we will use $X := L^2_{ad}([a,b]\times\Omega)$ to denote the space of all stochastic process $f(t,\omega), a \le t \le b, \omega \in \Omega$ satisfying the following conditions:*

(1)$f(t,\omega)$ is adapted to the filtration $\mathcal{F}_t$;

(2)$\int_a^b E(|f(t)|^2)dt < +\infty$.

Now we want to solve the stochastic integral equation:

$$x(t;w) = h(t_0;w) + \int_a^t \sigma(s,x(s;w))dB(s) + \int_a^t f(s,x(s;w))ds, \ a \le t \le b, \tag{4}$$

where:

(1)$\omega \in \Omega$, where Ω is the supporting set of the probability measure space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F}$ being the σ -algebra and P the probability measure;

(2)$x(t;w)$ is the unknown random variable for each $t \in [a,b]$;

(3)$h(t_0;w)$ is the knowm random variable and $E|h|^2 < +\infty$;

(4)$B(t)$ be a Brownian motion and $\{\mathcal{F}_t; a \le t \le b\}$ be a filtration so there $B(t)$ is $\mathcal{F}_t$ -measurable for each t and $B(t) - B(s)$ is independent of $\mathcal{F}_s$ for any $s < t$.

Let

$$Q = \{(t, X_t) \in R^2;\ t \in [a,b],\ and\ \|X_t\| \le r\ for\ fixed\ r > 0\}.$$

Theorem 1.5 *Assume the following conditions:*

(a) $f(s,X_s)$, $\sigma(s,X_s)$ are measurable on $[a,b]\times\Omega$;

$f(s,X_s) : Q \longrightarrow R$ is continuous;

$\sigma(s,X_s) : Q \longrightarrow R$ is continuous;

(b) We define

$$d = \sup_{(s,X_s)\in Q} \{\|f(s,X_s)\|, \|\sigma(s,X_s)\|\}, \tag{5}$$

and let the real number $a,\ b,\ d$ and random variable h be given that

$$3E[h^2]+3(1+b-a)(b-a)d^2 \le r^2. \tag{6}$$

(c) We set $X := L^2_{ad}([a,b]\times\Omega)$ (see Notation 2.1) and $M := \{X_t \in X; \|X_t\| \le r\}$.

Then the stochastic integral equation (4) has at least one solution $X_t \in M$.

Theorem 1.6 *Assume the following conditions:*

(a) $f(s,X_s),\ \sigma(s,X_s)$ are measurable on $[a,b]\times\Omega$;

(b)

$$\begin{aligned} |f(s,X_s)-f(s,Y_s)| &\le k_1|X_s-Y_s|; \qquad (7)\\ |\sigma(s,X_s)-\sigma(s,Y_s)| &\le k_2|X_s-Y_s|; \qquad (8)\end{aligned}$$

(c) Let the real number $a,\ b,\ c=\{k_1,k_2\}$ be given such that

$$0 \le 2c^2(1+b-a)(b-a) < 1. \tag{9}$$

(d) We set $X := L^2_{ad}([a,b]\times\Omega)$ (see Notation 2.1) and $M := \{X_t \in X; \|X_t\| \le r\}$.

Then the stochastic integral equation (4) has a unique solution $X_t \in M$.

2 Some Lemmas

We require the following Lemmas for proving the existence of the stochastic integral equation.

Lemma 2.1([1]) *Let X and Y be normed spaces over field K, and let*

$$A : M \subseteq X \longrightarrow Y$$

be a continuous operator on the compact nonempty subset M of X. Then, A is uniformly continuous on M.

Lemma 2.2([1]) *Let $X := L^2_{ad}([a,b]\times\Omega)$ with $\|X_t\| := (E|X_t|^2)^{\frac{1}{2}}$ and $-\infty < a < b < +\infty$. Suppose that we are given a set M in X such that*

(1)M is bounded , i.e., $\|X_t\| \le r$ for all $X_t \in M$ and fixed $r \ge 0$.

(2)M is equicontinuous, i.e., for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|t_1-t_2| < \delta \quad \text{and} \quad X_t \in M \quad \text{imply} \quad |X_{t_1}-X_{t_2}| < \varepsilon.$$

Then, M is a relatively compact subset of X.

Definition 2.1([1]) *Let X and Y be normed spaces over field K. The operator*

$$A : M \subseteq X \longrightarrow Y$$

is called compact iff

(1) A is continuous, and

(2) A transforms bounded sets into relatively compact sets.

Lemma 2.3([3]) *(Itô Isometry)*

For each $X_t, Y_t \in L^2_{ad}([a,b]\times\Omega)$*, we have*

$$E[(\int_a^b f(t,w)dB(t))^2] = E[\int_a^b f^2(t,w)dt]$$

3 The Proof of Theorem 1.5

Proof: We divide the proof into three steps:

Step 1: We prove that $M = \{X_t \in X; \|X_t\| \leq r\}$ is closed, convex subset of $L^2_{ad}([a,b]\times\Omega)$ (see Notation 2.1).

(A) We prove M is closed.

Let $X_t^{(n)} \in M$ for all n, i.e.,

$$\|X_t^{(n)}\| \leq r \quad for\ all\ n.$$

If $X_t^{(n)} \longrightarrow X_t$ as $n \longrightarrow +\infty$, then $\|X_t\| \leq r$, and hence $X_t \in M$.

(B) We prove M is convex.

If $X_t, Y_t \in M$ and $0 \leq \alpha \leq 1$, then

$$\begin{aligned}\|\alpha X_t + (1-\alpha)Y_t\| &\leq \|\alpha X_t\| + \|(1-\alpha)Y_t\| \\ &\leq \alpha r + (1-\alpha) r \\ &= r\end{aligned}$$

Hence

$$\alpha X_t + (1-\alpha)Y_t \in M.$$

Step 2: We prove that $A : M \longrightarrow M$ is a compact operator.

Define $A : M \longrightarrow M$

$$A(X_t) = h(t_0; w) + \int_a^t \sigma(s, x(s;w))dB(s) + \int_a^t f(s, x(s;w))ds,\ a \leq t \leq b, \tag{10}$$

Then

(a) $A : M \longrightarrow M$ is a continuous operator.

By Lemma 2.1, we know $f(s, X_s), \sigma(s, X_s)$ are uniformly continuous on the compact set Q. This implies that, for each $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\|\sigma(s, X_s) - \sigma(s, Y_s)\| < \varepsilon_1$$

$$\|f(s,X_s) - f(s,Y_s)\| < \varepsilon_2$$

for all$(s,X_s),(s,Y_s) \in Q$ with $\|X_s - Y_s\| < \delta$.

For each $X_t, Y_t \in M$, we have

$$\begin{aligned}
\|AX_t - AY_t\|^2 &= E(\int_a^t (\sigma(s,X_s) - \sigma(s,Y_s))dB(s) \\
&+ \int_a^t (f(s,X_s) - f(s,Y_s))ds)^2
\end{aligned}$$

Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ to get

$$\begin{aligned}
\|AX_t - AY_t\|^2 &\leq 2E(\int_a^t (\sigma(s,X_s) - \sigma(s,Y_s))dB(s))^2 \\
&+ 2E(\int_a^t (f(s,X_s) - f(s,Y_s))ds)^2 \qquad (11)
\end{aligned}$$

Applying the Itô Isometry to $E(\int_a^t(\sigma(s,X_s) - \sigma(s,Y_s))dB(s))^2$, we get:

$$\begin{aligned}
E(\int_a^t (\sigma(s,X_s) - \sigma(s,Y_s))dB(s))^2 &= E(\int_a^t (\sigma(s,X_s) - \sigma(s,Y_s))^2 ds) \\
&= \int_a^t E(\sigma(s,X_s) - \sigma(s,Y_s))^2 ds \\
&= \int_a^t \|\sigma(s,X_s) - \sigma(s,Y_s)\|^2 ds \\
&< (b-a)\varepsilon_1^2 \qquad (12)
\end{aligned}$$

For $E(\int_a^t(f(s,X_s) - f(s,Y_s))ds)^2$, we use Schwarz's inequality to get

$$\begin{aligned}
E(\int_a^t (f(s,X_s) - f(s,Y_s))ds)^2 &\leq E((t-a)\int_a^t (f(s,X_s) - f(s,Y_s))^2 ds) \\
&\leq (b-a)\int_a^t E(f(s,X_s) - f(s,Y_s))^2 ds \\
&= (b-a)\int_a^t \|f(s,X_s) - f(s,Y_s)\|^2 ds \\
&< (b-a)^2\varepsilon_2^2 \qquad (13)
\end{aligned}$$

Put equations (12) and (13) into equation (11) to get

$$\begin{aligned}
\|AX_t - AY_t\|^2 &< 2(b-a)[(b-a)\varepsilon_2^2 + \varepsilon_1^2] \\
&= \varepsilon^2
\end{aligned}$$

Therefore for each $X_t, Y_t \in M$, there exists $\delta > 0$, when $\|X_t - Y_t\| \leq \delta$, we have

$$\|AX_t - AY_t\| < \varepsilon.$$

That is: A is a continuous operator.

(b) A(M) is bounded.

For each $X_t \in M$

$$\begin{aligned}
\|AX_t\|^2 &= E(h(t_0;\omega) + \int_a^t \sigma(s, X_s)dB(s) + \int_a^t f(s, X_s)ds)^2 \\
&\leq 3E[h^2] + 3E(\int_a^t \sigma(s, X_s)dB(s))^2 + 3E(\int_a^t f(s, X_s)ds)^2 \\
&\leq 3E[h^2] + 3\int_a^t E|\sigma(s, X_s)|^2 ds + 3(b-a)\int_a^t E|f(s, X_s)|^2 ds \\
&= 3E[h^2] + 3\int_a^t \|\sigma(s, X_s)\|^2 ds + 3(b-a)\int_a^t \|f(s, X_s)\|^2 ds \\
&\leq 3E[h^2] + 3(b-a)(1+b-a)d^2 \\
&\leq r^2 \qquad (14)
\end{aligned}$$

Thus A(M) is bounded.

(c) A(M) is equicontinuous.

For each $X_t \in M$, we have

$$\begin{aligned}
\|AX_{t_1} - AX_{t_2}\|^2 &= E(\int_{t_2}^{t_1} \sigma(s, X_s)dB(s) + \int_{t_2}^{t_1} f(s, X_s)ds)^2 \\
&\leq 2E(\int_{t_2}^{t_1} \sigma(s, X_s)dB(s))^2 + 2E(\int_{t_2}^{t_1} f(s, X_s)ds)^2 \\
&\leq 2\int_{t_2}^{t_1} E|\sigma(s, X_s)|^2 ds + 2(b-a)\int_{t_2}^{t_1} E|f(s, X_s)|^2 ds \\
&\leq 2(1+b-a)|t_1 - t_2| d^2
\end{aligned}$$

Take

$$\delta = \frac{\varepsilon^2}{2(1+b-a)d^2},$$

then for each $\varepsilon > 0$, there exists

$$\delta = \frac{\varepsilon^2}{2(1+b-a)d^2},$$

when $|t_1 - t_2| < \delta$, we have

$$\|AX_{t_1} - AX_{t_2}\| < \varepsilon.$$

Hence A(M) is equicontinuous.

Then by Lemma 2.2 and Definition 2.1, we know $A : M \longrightarrow M$ is a compact operator.

Step 3: we prove that $A(M) \subseteq M$.

For each $X_t \in M$, we have

$$\begin{aligned}
\|AX_t\|^2 &= E(h + \int_a^t \sigma(s, X_s)dB(s) + \int_a^t f(s, X_s)ds)^2 \\
&\leq 3E[h^2] + 3E(\int_a^t \sigma(s, X_s)dB(s))^2 + 3E(\int_a^t f(s, X_s)ds)^2 \\
&\leq 3E[h^2] + 3\int_a^t E|\sigma(s, X_s)|^2 ds + 3(b-a)\int_a^t E|f(s, X_s)|^2 ds
\end{aligned}$$

where $f(s,X_s), \sigma(s,X_s) \in L^2_{ad}([a,b]\times\Omega)$.

So

$$\int_a^b \|AX_t\|^2 dt < +\infty,$$

that is

$$AX_t \in L^2_{ad}([a,b]\times\Omega),$$

meanwhile, we have proved $\|AX_t\| \leq r$ in (14), therefore $AX_t \in M$, that is $A(M) \subseteq M$.

Thus the Schauder's Fixed Point Theorem tells us that equation (4) has at least one solution $X_t \in M$.

4 The Proof of the Theorem 1.6

Proof: In the proof of Theorem 1.5, we have proved that M is closed.

We now show that A is a contractive mapping:

For each $X_t, Y_t \in M$, we have

$$\begin{aligned}
\|AX_t - AY_t\|^2 &= E\Big(\int_a^t (\sigma(s,X_s)-\sigma(s,Y_s))dB(s) + \int_a^t (f(s,X_s)-f(s,Y_s))ds\Big)^2 \\
&\leq 2E\Big(\int_a^t (\sigma(s,X_s)-\sigma(s,Y_s))dB(s)\Big)^2 + 2E\Big(\int_a^t (f(s,X_s)-f(s,Y_s))ds\Big)^2 \\
&\leq 2\int_a^t E(\sigma(s,X_s)-\sigma(s,Y_s))^2 ds + 2(b-a)\int_a^t E(f(s,X_s)-f(s,Y_s))^2 ds \\
&\leq 2k_2^2\int_a^t E|X_s-Y_s|^2 ds + 2k_1^2(b-a)\int_a^t E|X_s-Y_s|^2 ds \\
&\leq 2c^2(1+b-a)(b-a)\|X_t-Y_t\|^2
\end{aligned}$$

Let

$$k^2 = 2c^2(1+b-a)(b-a) < 1,$$

then

$$\|AX_t - AY_t\| \leq k\|X_t - Y_t\|, \quad 0 \leq k < 1,$$

Therefore A is k-contractive.

Then the Banach's fixed point theorem tells us that the stochastic integral equation (4) has a unique solution $X_t \in M$.

5 Comparing Theorem 1.5 with Theorem 1.6

Comparing Theorem 1.5 with Theorem 1.6 we know when we use Schauder's fixed point theorem to prove the existence of the solution for the integral equation, we need conditions (5) and (6).

But when we use Banach's fixed point theorem to prove the existence of the solution for the stochastic integral equation, we need conditions (7), (8) and (9).

Obviously, the condition (6) is weaker than the condition (9).

6 An Example for Theorem 1.5

We apply the above Theorem 1.5 to the linear stochastic integral equation:

$$X_t = \int_0^t f(s)X_s ds + \int_0^t g(s)X_s dB(s), \quad 0 \le t \le 1, \tag{15}$$

Proof: Define the operator:

$$A(X_t) = \int_0^t f(s)X_s ds + \int_0^t g(s)X_s dB(s), \quad 0 \le t \le 1,$$

Obviously, $f(s)X_s$ and $g(s)X_s$ are measurable and continuous. We define that $d = \sup E|X_s|^2$, $c = \max\{f(s)^2, g(s)^2\}$, $6cd = r^2$ and set $X := L^2_{ad}([a,b] \times \Omega)$ and $M = \{X_t \in X; \|X_t\| \le r\}$.

Then all conditions of Theorem 1.5 hold, so (15) has at least one solution.

Especially, if we let $f(s) = u, g(s) = \sigma$, then the equation is the geometric Brownian motion equation

$$X_t = \int_0^t uX_s ds + \int_0^t \sigma X_s dB(s), \quad 0 \le t \le 1, \tag{16}$$

where u is the expected return rate(constant), σ is volatility(constant), $B(t)$ is standard Brown motion.

We can easily to get that the equation (16) has at least one solution. It well known that the existence of the solution of the equation (16) is important to the financial Black-Scholes model .

Acknowledgements. We would like to thank our teacher Processor Zhang Shiqing for his lectures on Functional Analysis, meanwhile We would like to thank him for organizing the seminar on financial mathematics and his many helpful discussions, suggestions and corrections about this paper.

References

[1] E. Zeidler, Applied Functional Analysics: Applications to Mathematical Physics, Spring-Verlag, New York-Berlin, 1995, P. 18-64.

[2] Hui-Hsiung Kuo, Introduction to Stochastic Integration, Springer, 2006, P. 43-61.

[3] Bernt ϕ Ksendl,Stochastic Differential Equations: An introduction with Applications, Springer-Verlag, 2005, P.29.

[4] Bharucha-Reid, A. P., Random Integral Equations, Academic Press, New York, 1972.

[5] Doob, J. Stochastic Processes, Wiley, New York, 1953.

[6] Padgett, W. J. and Tsokos, C. P., On a stochastic integral of the Volterra type in telephone traffic theory, J. Appl. Prob. 8, 1971, 269-275.

[7] Tsokos, C. P. and Padgett, W. J., Random Integral Equations with Application to Life Sciences and Engineering, Academic Press, New York, 1974.

[8] Rao, A. N. V. and Tsokos, C. P., On a class of stochastic integral equation, Coll. Math. 35, 1976, 141-146.